TASI Lectures: Cosmology III

Numbers and Defintions

1. Apparent magnitude $m \equiv -2.5 \log_{10}(F) +$ constant, where F is flux
2. Absolute magnitude $M = 4.76 + 2.5 \log_{10}(L/L_{\odot})$, w/ L the luminosity and $L_{\odot} = 3.826 \times 10^{33}$ ergs/sec is the sun's luminosity.
3. Since the flux scales as the luminosity divided by distance squared,
$$m - M = 5 \log_{10}(d_L/10\,\mathrm{pc}).$$
4. Comoving distance to scale factor a in a flat universe is $\chi(a) = \int_a^1 da'/(a'^2 H(a'))$.
5. Comoving Horizon, $\eta(a) \equiv \int_0^a da'/(a'^2 H(a'))$.
6. Luminosity distance in a flat universe,
$$d_L(a) = \chi(a)/a = \chi(z) \times (1+z) = (1+z) \int_0^z dz'/H(z').$$
7. Comoving Hubble Radius, $(aH)^{-1}$.
8. Fourier convention:
$$\tilde{f}(\vec{k}) = \int d^3x e^{-i\vec{k}\cdot\vec{x}} f(\vec{x}).$$
Both $\vec{k}$ and $\vec{x}$ are *comoving* so do not change as the universe expands.
9. Power spectrum:
$$\langle \tilde{\delta}(\vec{k})\tilde{\delta}(\vec{k}')\rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P(k).$$
Here the angular brackets denote averages over all possible realizations. I.e., a given $k-$mode has its amplitude drawn from a Gaussian distribution with a variance given by the power spectrum. The $\delta^3()$ on the right hand side is the Dirac delta function.

Results

1. Power spectrum of scalar perturbations produced during slow roll inflation with a single inflaton field:
$$P_\Phi(k) = \frac{2}{9} \frac{(8\pi G)^2}{k^3} \frac{H^2}{(V'/V)^2}\Big|_{aH=k} \equiv A_S k^{n_s - 4}.$$
2. Tilt in slow-roll inflation with a single inflaton field:
$$n_s = 1 - \frac{3}{8\pi G}\left(\frac{V'}{V}\right)^2 + \frac{2}{8\pi G}\frac{V''}{V}.$$

Exercises

1. The *distance modulus* μ is defined as $m - M$. Plot the distance modulus as a function of redshift in a flat, matter dominated universe $\Omega_m = 1$. The requisite integral can be done analytically in this case. Then plot μ when $\Omega_m = 0.26$ and $\Omega_{\rm de} = \Omega_\Lambda = 0.74$. For this you need to compute the integral numerically. [Once you have the code running, it might fun to consider other dark energy models, with a variety of equations of state.] Which curve does SN1997ap (with $m = 24.32$ at $z = 0.83$) come closer to? Use SN1992 (at $z = 0.026$ with $m = 16.08$) to determine M.

2. Show that the conformal time η scales as $a^{1/2}$ in a matter dominated universe and as a in one dominated by radiation. Show that in a universe with only matter and radiation [i.e. probably our universe at redshifts earlier than one],

$$\eta = \frac{2}{\sqrt{\Omega_m h^2}} \left[\sqrt{a + a_{eq}} - \sqrt{a_{eq}}\right]$$

where a_{eq} is the epoch at which the matter density is equal to the radiation density.

3. It is convenient in many models of inflation to define a slow roll parameter $\epsilon \equiv d(H^{-1})/dt$. As you might expect, since H is roughly constant, ϵ is typically small throughout inflation. In fact, one definition of the end of inflation is the epoch at which $\epsilon = 1$. Use the equation of motion for a scalar field in an expanding universe

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + V' = 0$$

in the slow-roll limit (where the second derivative is much smaller than H times the first derivative) to derive an expression relating ϵ to the inflaton potential and its derivatives:

$$\epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 .$$

3. Determine the predictions of an inflationary model with a quartic potential,

$$V(\phi) = \lambda\phi^4.$$

(a) Compute the slow roll parameter ϵ in terms of ϕ.

(b) Determine ϕ_e, the value of the field at which inflation ends, by setting $\epsilon = 1$ at the end of inflation.

(c) Find the value of ϕ when the mode $k = a_0 H_0$ leaves the horizon during inflation. To do this, assume 60 e-folds. That is, assume that the universe inflates by a factor of e^{60} between the time when this mode exits the horizon and time at which inflation ends. Rewrite

$$N = \int_t^{t_e} dt' H(t')$$

as an integral over ϕ to determine ϕ at horizon exit. Show that this mode leaves the horizon when $\phi^2 \simeq 60 m_{\text{Planck}}^2/\pi$.

(e) Determine the predicted value of n_s.

(f) Estimate the scalar amplitude in terms of λ. As a rough estimate, assume that $k^3 P_\Phi(k)$ for this mode is equal to 10^{-8}. What value does this imply for λ?

This model illustrates many of the features of contemporary models. In it, (i) the field is of order – even greater than – the Planck scale, but (ii) the energy scale V is much smaller because of (iii) the very small coupling constant.